Today: last lecture: Coulomb branch.

Classical version of Coulomb branch:

$$T^*\check{G}/\!\!/\!\!/\check{G} \underset{\uparrow}{\cong} T^*\check{H}/W, \quad \check{H} \subseteq \check{G} \text{ max torus.}$$

probably known only if LHS is reduced.

+ quantum corrections.

<u>Data</u>: G = reductive group

V = symplectic G-rep ($G \to Sp(V)$).

$\rightsquigarrow$ Higgs branch: $V/\!\!/\!\!/G$.

IIB string theory: explain 5-branes, 3-branes pertaining to type A quivers.

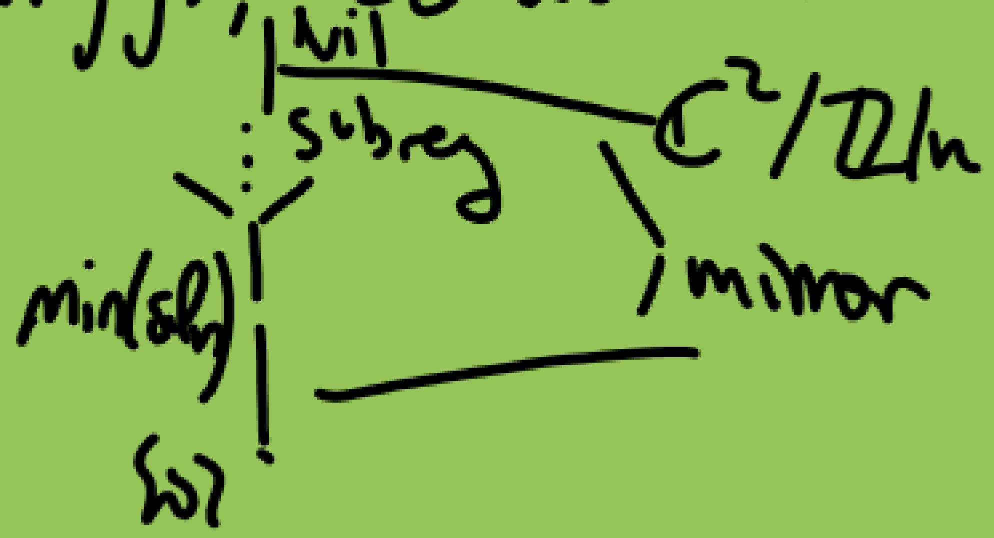
(quiver: Higgs branch = Nakajima q. var.)

→ 2 components of moduli space of vacua

vacuum: possible physical state, no matter.

Higgs, Coulomb. Ex: Both = nil($\mathfrak{sl}_n$).

(SD:



Along subreg in Coulomb, have another Higgs of min($\mathfrak{sl}_n$).

That was explanation of "mixed components"
One sticks to Coulomb + Higgs mostly.

x_3, x_4, x_5
$(+ x_1, x_2)$ = 5-dim "brane"
5-brane.

x_6 $(+ x_1, x_2)$: 3 brane

X : x_7, x_8, x_9 $(+ x_1, x_2)$: 5-brane.

X $\longleftrightarrow$ | : rotation : interchanges types of 5-branes.

$(G, V) \rightsquigarrow$ moduli of <u>brane configurations</u>:

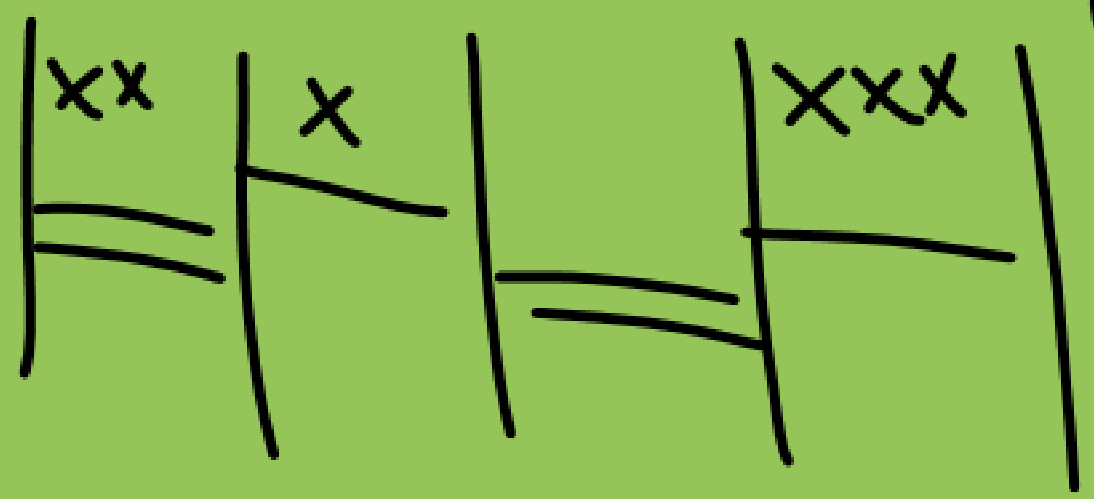
type A gauge

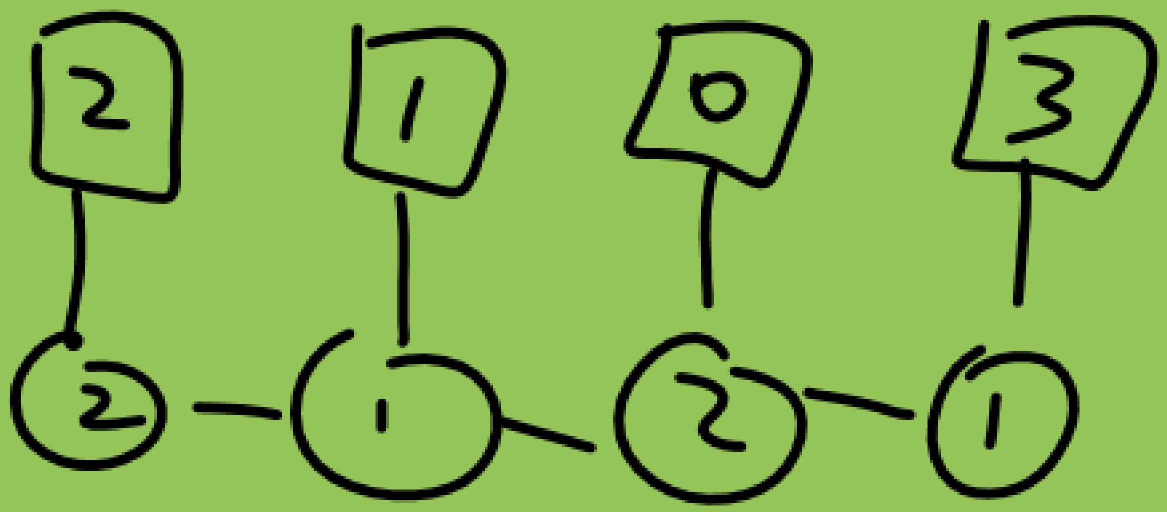
move these branes, put data on etc.

Rotation interchanges Higgs, Coulomb.

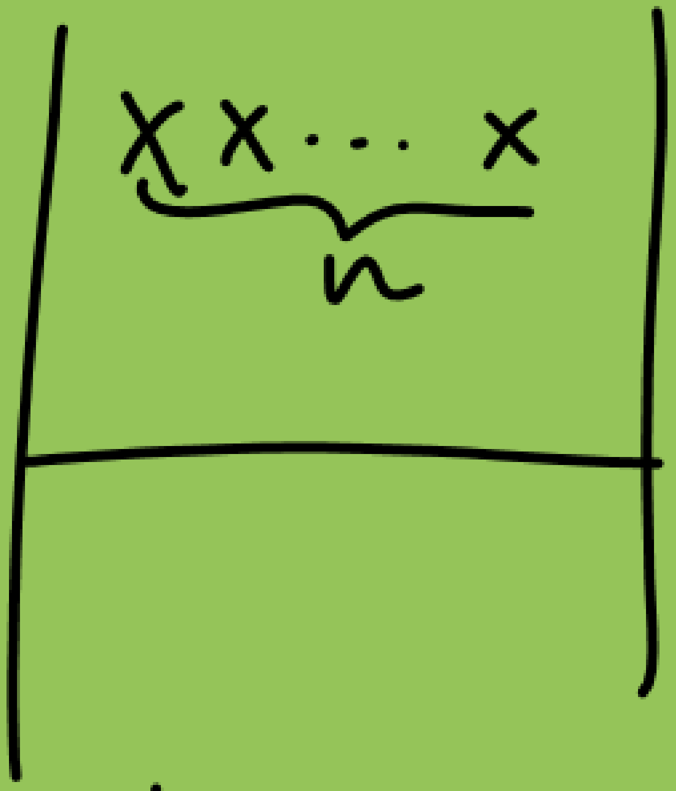
Relation to quivers:

type A





Example of mirror quivers:



$\leftrightarrow$

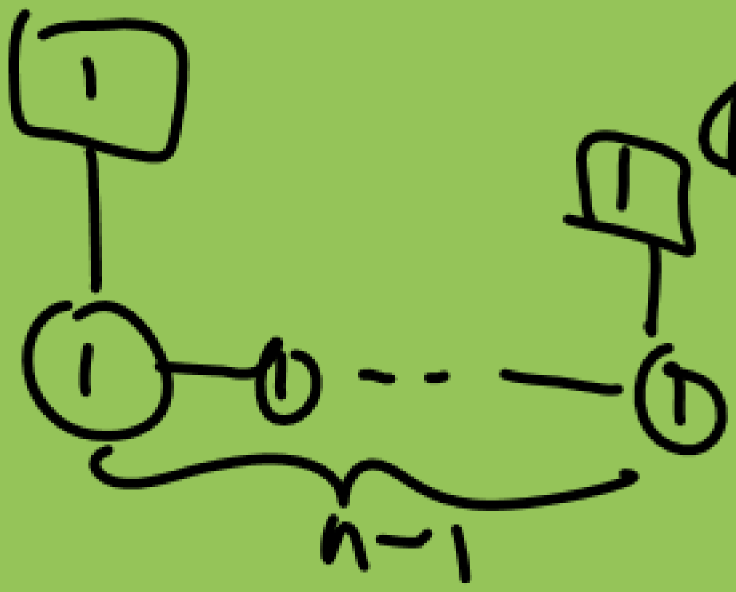
Higgs: $T^*\mathbb{C}^n /\!/\!/ \mathbb{C}^\times$

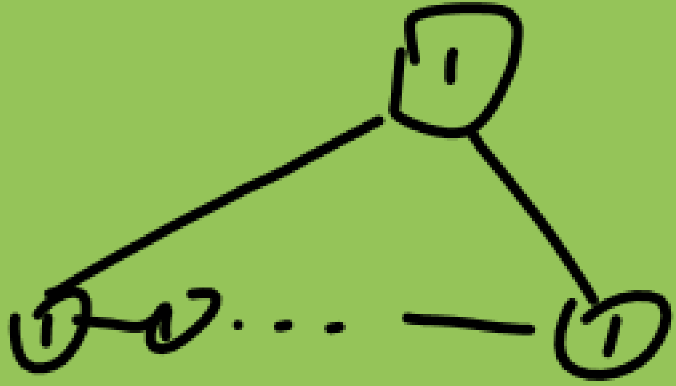
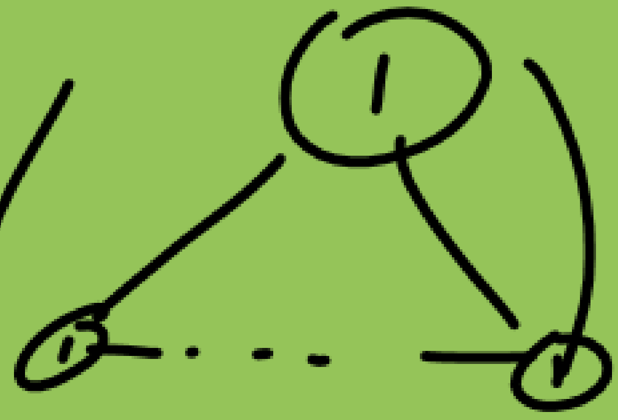
$= \min(\mathfrak{sl}_n)$

Coulomb: Slodowy to subreg:

$\mathbb{C}^2/\mathbb{Z}_n$.

Rotate:



Higgs:  $\cong$ Higgs()

$\cong \mathbb{C}^2/\mathbb{Z}_n$.

Coulomb: min(ℓ, n)

$H($  $) \cong C($  $)$

$C($ " $) \cong H($ " $)$

Quivers are "mirror".

3D mirror symm.

Physics says: If you have quiver Q,
dim vector + framing dim,

$I \subseteq Q_0 :=$ vertices subset of <u>balanced</u> nodes

$\Rightarrow$ $\underset{\text{Lie.}}{\mathfrak{g}_{Q|_I}} \subseteq \left(\mathcal{O}(M_C), \{-,-\}\right)$

$$\Rightarrow \text{ in } \mathcal{C}\left(\begin{tikzcd} \boxed{1} \arrow[d, dash] & & \boxed{1} \arrow[d, dash] \\ 1 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 \end{tikzcd}\right), \; M_C \text{ has } \mathfrak{sl}_n \text{ symmetry.}$$

$$\cong \; \min(\mathfrak{sl}_n) \circlearrowleft SL_n.$$

How? <u>Monopole fla</u>

$$H(\mathcal{O}(M_c);t) := \sum_{n\geq 0} \dim \mathcal{O}(M_c)_n t^n$$

"good" or "ugly" cone

$$= \sum_{\substack{\text{dom. coweights} \\ \lambda}} t^{2\Delta(\lambda)} P_G(t,\lambda), \quad P_G(t,\lambda) = \prod \frac{1}{(1-t^{2d_i})}$$

d_i = degs of gens of $\mathbb{C}[\mathfrak{g}^*]^{W_\lambda}$

W_λ = Weyl of $Z_G(\lambda(\mathbb{C}^\times))$.

$$\Delta(\lambda) = -\sum_{\alpha \in \Phi^+_G} |\alpha(\lambda)| + \frac{1}{4} \sum_{\mu \text{ weight.}} \mu(\lambda) \dim V_\mu$$

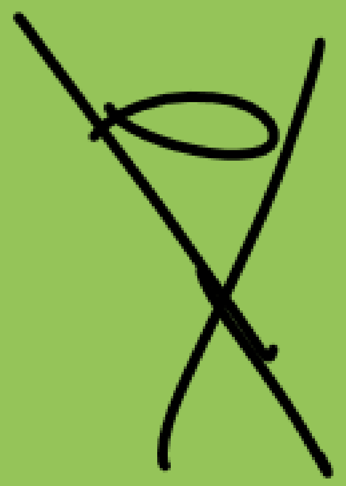
G, V "good" if $\Delta(\lambda) \geq 1 \quad \forall \lambda \neq 0$

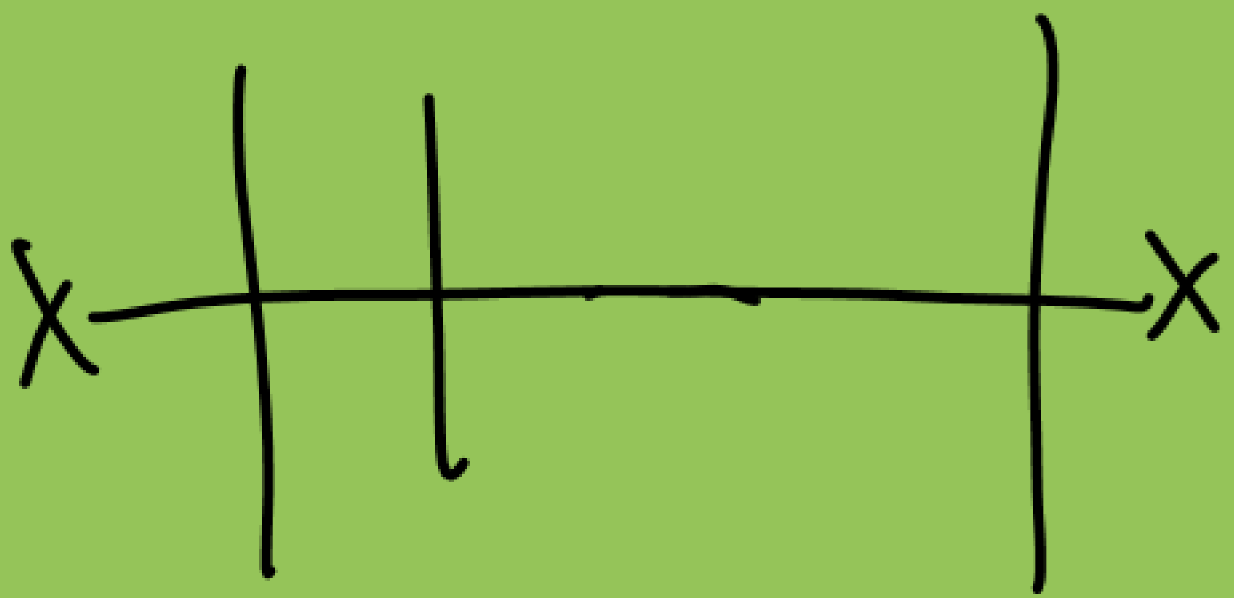
"ugly" if $\Delta(\lambda) \geq \frac{1}{2} \quad \forall \lambda \neq 0$ (+ not good)

"bad" otherwise.

"good": M_{cf} cone with unique cone point

"ugly": M_{cf} cone $\times \mathbb{C}^d$.





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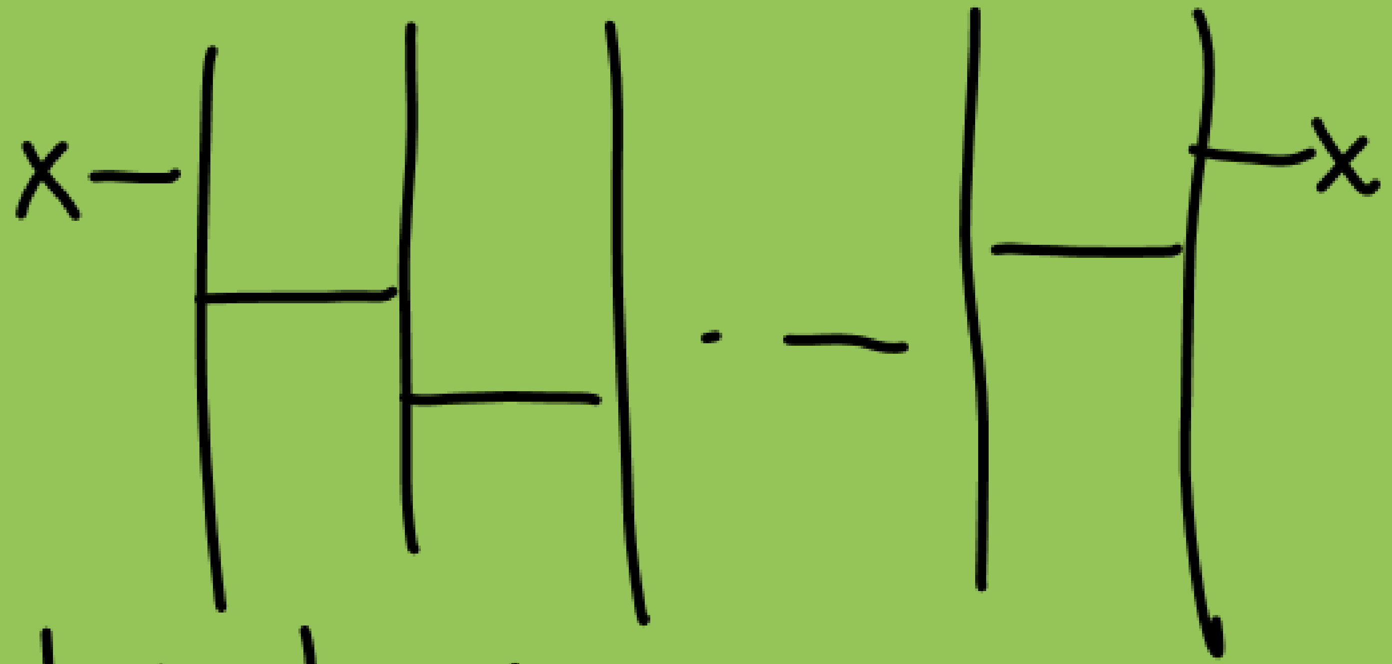
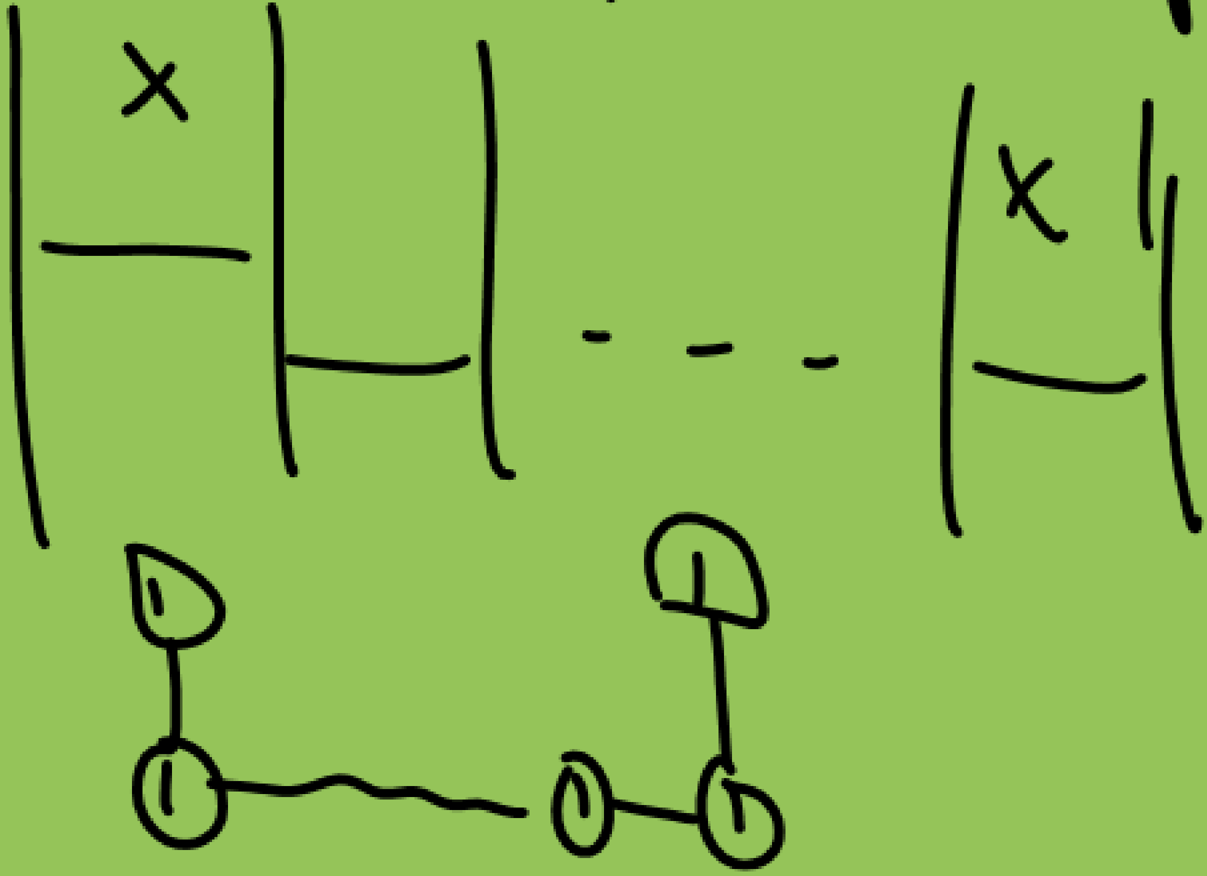




Borel-Moore homology:

version of homology for locally compact spaces (not compact)

X homotopic to finite CW cx

$$\Rightarrow H_*^{BM}(X) \cong H_*(X \cup \{\infty\}, \{\infty\})$$

1-pt compactification.

<u>Ex</u> $X = \mathbb{R}^n$: $H_*^{BM} = \begin{cases} \mathbb{Q}, & * = n \\ 0, & \text{otherwise} \end{cases}$

$G_K, G_\mathcal{O}, \quad N_K = N \otimes_{\mathbb{C}} K, \quad N_\mathcal{O} = N \otimes_{\mathbb{C}} \mathcal{O}$

$$\mathcal{O}(M_C) := H_*^{BM, G_K}\left(\mathcal{T} \times_{N_K} \mathcal{T}\right)$$

To make precise:

$N = 0: \mathcal{T} = G_K / G_\mathcal{O} =: Gr_G$ "Affine Gr."

$$Gr_G \times Gr_G \cong G_K \times_{G_\mathcal{O}} Gr_G \rightsquigarrow H_*^{BM, G_\mathcal{O}}(Gr_G)$$

$$(g\mathcal{O}, gx) \longleftarrow\!\shortmid (g, x)$$

$G_{\mathcal{O}}$ profinite $:= \varprojlim \underbrace{G_{\mathcal{O}}/G_{t^k\mathcal{O}}}_{\text{f.t.}}$

Gr_G = ind-finite:
inductive limit (union) of $G_{\mathcal{O}}$-orbits of finite type

Thm (BFM): $H_*^{BM, G_{\mathcal{O}}}(Gr_G) \cong$ "universal centralizer" of $G^\vee$

NOT core: "bad" theory.

It's a nicer version of: $x \in \mathfrak{g}^\vee$, take $Z_{G^\vee}(x)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

(jumps in size at $0 \in \mathfrak{g}^\vee$) $\qquad \{g \in G \mid \mathrm{Ad}_g x = x\}$

One wants: regular centralisers: restr. to $x \in \mathfrak{g}^{\vee \mathrm{reg}}$

Want: bundle over $\mathfrak{g}^\vee /\!/ G^\vee \cong \mathfrak{h}^\vee / W =: \mathfrak{c}^\vee$

__Kostant section__: $\kappa : \mathfrak{c}^\vee \hookrightarrow \mathfrak{g}^\vee$, target is regular, affine linear,

im $\kappa = e + \ker(\mathrm{ad}\, f)$, $\langle e, h, f \rangle \simeq \mathfrak{sl}_2$, e regular.

"J" = reg univ. centralizer = bundle of groups over $\mathfrak{c}$,

fiber at $x \in \mathfrak{c}$ is $Z_G k(x)$.

= affine symplectic var, $J \twoheadrightarrow \mathfrak{c}$ integrable system

abelian group fibers.

Subcase: $\mathcal{M}_C \cong \mathfrak{t} \times T^\vee$, actually <u>no</u> q. corrs

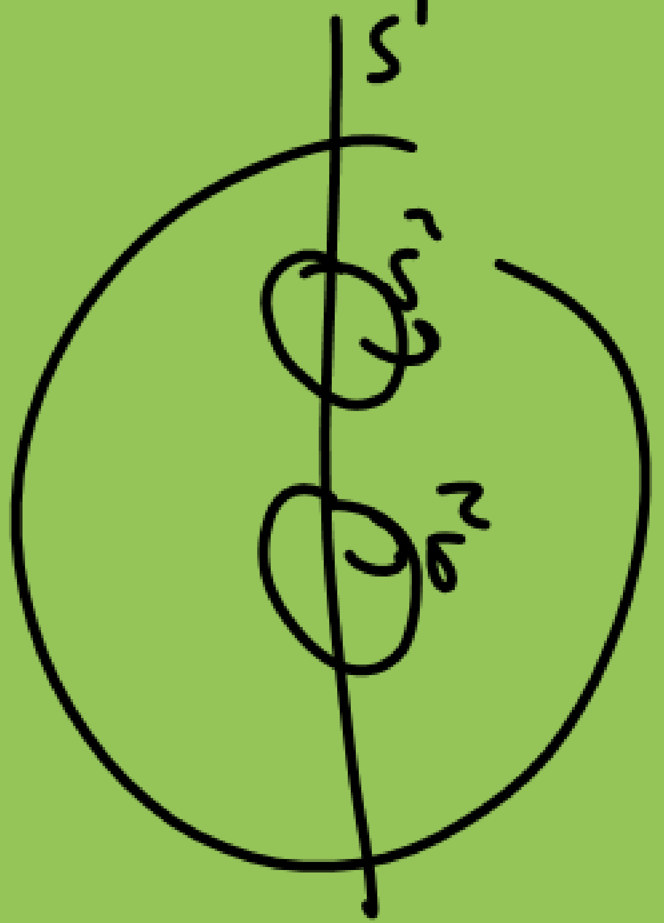
$$\mathfrak{t} \times T^\vee = T^*(T^\vee). \quad (/// \, T, \; T \text{ acts trivially})$$

General fix: $\mathcal{J} \times_{N_k} \mathcal{J} \cong G_k \times_{G_O} \mathcal{R} = \dfrac{G_k \times \mathcal{R}}{G_O}$.

fiber prod

replacing Gr_G

$$\mathcal{R} = \left\{ [g, n]_{G_O} \in \frac{G_k \times N_O}{G_O} \;\middle|\; n \in g^{-1} N_O \right\}$$



$S^2 \subseteq S^3$

Get E_1 alg, but with equivariant param from S^1

$(\hbar)$: quantisation.